Bose–Einstein Condensation in Dilute Gases

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne,VIC 3207, Australia
Ruiz de Alarcón 13, 28014, Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Computer Modern 11/14pt. *System* LaTeX 2_ε [DBD]

A catalogue record of this book is available from the British Library

Library of Congress Cataloguing in Publication Data

Pethick, Christopher.
Bose–Einstein condensation in dilute gases / C. J. Pethick, H. Smith.
p. cm.
Includes bibliographical references and index.
ISBN 0 521 66194 3 – ISBN 0 521 66580 9 (pb.)
1. Bose–Einstein condensation. I. Smith, H. 1939– II. Title.
QC175.47.B65 P48 2001
530.4′2–dc21 2001025622

ISBN 0 521 66194 3 hardback
ISBN 0 521 66580 9 paperback

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Introduction

Bose–Einstein condensates in dilute atomic gases, which were first realized experimentally in 1995 for rubidium [1], sodium [2], and lithium [3], provide unique opportunities for exploring quantum phenomena on a macroscopic scale.[1] These systems differ from ordinary gases, liquids, and solids in a number of respects, as we shall now illustrate by giving typical values of some physical quantities.

The particle density at the centre of a Bose–Einstein condensed atomic cloud is typically 10^{13}–10^{15} cm^{-3}. By contrast, the density of molecules in air at room temperature and atmospheric pressure is about 10^{19} cm^{-3}. In liquids and solids the density of atoms is of order 10^{22} cm^{-3}, while the density of nucleons in atomic nuclei is about 10^{38} cm^{-3}.

To observe quantum phenomena in such low-density systems, the temperature must be of order 10^{-5} K or less. This may be contrasted with the temperatures at which quantum phenomena occur in solids and liquids. In solids, quantum effects become strong for electrons in metals below the Fermi temperature, which is typically 10^4–10^5 K, and for phonons below the Debye temperature, which is typically of order 10^2 K. For the helium liquids, the temperatures required for observing quantum phenomena are of order 1 K. Due to the much higher particle density in atomic nuclei, the corresponding degeneracy temperature is about 10^{11} K.

The path that led in 1995 to the first realization of Bose–Einstein condensation in dilute gases exploited the powerful methods developed over the past quarter of a century for cooling alkali metal atoms by using lasers. Since laser cooling alone cannot produce sufficiently high densities and low temperatures for condensation, it is followed by an evaporative cooling stage, in

[1] Numbers in square brackets are references, to be found at the end of each chapter.

which the more energetic atoms are removed from the trap, thereby cooling the remaining atoms.

Cold gas clouds have many advantages for investigations of quantum phenomena. A major one is that in the Bose–Einstein condensate, essentially all atoms occupy the same quantum state, and the condensate may be described very well in terms of a mean-field theory similar to the Hartree–Fock theory for atoms. This is in marked contrast to liquid ^{4}He, for which a mean-field approach is inapplicable due to the strong correlations induced by the interaction between the atoms. Although the gases are dilute, interactions play an important role because temperatures are so low, and they give rise to collective phenomena related to those observed in solids, quantum liquids, and nuclei. Experimentally the systems are attractive ones to work with, since they may be manipulated by the use of lasers and magnetic fields. In addition, interactions between atoms may be varied either by using different atomic species, or, for species that have a Feshbach resonance, by changing the strength of an applied magnetic or electric field. A further advantage is that, because of the low density, 'microscopic' length scales are so large that the structure of the condensate wave function may be investigated directly by optical means. Finally, real collision processes play little role, and therefore these systems are ideal for studies of interference phenomena and atom optics.

The theoretical prediction of Bose–Einstein condensation dates back more than 75 years. Following the work of Bose on the statistics of photons [4], Einstein considered a gas of non-interacting, massive bosons, and concluded that, below a certain temperature, a finite fraction of the total number of particles would occupy the lowest-energy single-particle state [5]. In 1938 Fritz London suggested the connection between the superfluidity of liquid ^{4}He and Bose–Einstein condensation [6]. Superfluid liquid ^{4}He is the prototype Bose–Einstein condensate, and it has played a unique role in the development of physical concepts. However, the interaction between helium atoms is strong, and this reduces the number of atoms in the zero-momentum state even at absolute zero. Consequently it is difficult to measure directly the occupancy of the zero-momentum state. It has been investigated experimentally by neutron scattering measurements of the structure factor at large momentum transfers [7], and the measurements are consistent with a relative occupation of the zero-momentum state of about 0.1 at saturated vapour pressure and about 0.05 near the melting curve [8].

The fact that interactions in liquid helium reduce dramatically the occupancy of the lowest single-particle state led to the search for weakly-interacting Bose gases with a higher condensate fraction. The difficulty with most substances is that at low temperatures they do not remain gaseous, but form solids, or, in the case of the helium isotopes, liquids, and the effects of interaction thus become large. In other examples atoms first combine to form molecules, which subsequently solidify. As long ago as in 1959 Hecht [9] argued that spin-polarized hydrogen would be a good candidate for a weakly-interacting Bose gas. The attractive interaction between two hydrogen atoms with their electronic spins aligned was then estimated to be so weak that there would be no bound state. Thus a gas of hydrogen atoms in a magnetic field would be stable against formation of molecules and, moreover, would not form a liquid, but remain a gas to arbitrarily low temperatures.

Hecht's paper was before its time and received little attention, but his conclusions were confirmed by Stwalley and Nosanow [10] in 1976, when improved information about interactions between spin-aligned hydrogen atoms was available. These authors also argued that because of interatomic interactions the system would be a superfluid as well as being Bose–Einstein condensed. This latter paper stimulated the quest to realize Bose–Einstein condensation in atomic hydrogen. Initial experimental attempts used a high magnetic field gradient to force hydrogen atoms against a cryogenically cooled surface. In the lowest-energy spin state of the hydrogen atom, the electron spin is aligned opposite the direction of the magnetic field ($H\downarrow$), since then the magnetic moment is in the same direction as the field. Spin-polarized hydrogen was first stabilized by Silvera and Walraven [11]. Interactions of hydrogen with the surface limited the densities achieved in the early experiments, and this prompted the Massachusetts Institute of Technology (MIT) group led by Greytak and Kleppner to develop methods for trapping atoms purely magnetically. In a current-free region, it is impossible to create a local maximum in the magnitude of the magnetic field. To trap atoms by the Zeeman effect it is therefore necessary to work with a state of hydrogen in which the electronic spin is polarized parallel to the magnetic field ($H\uparrow$). Among the techniques developed by this group is that of evaporative cooling of magnetically trapped gases, which has been used as the final stage in all experiments to date to produce a gaseous Bose–Einstein condensate. Since laser cooling is not feasible for hydrogen, the gas is precooled cryogenically. After more than two decades of heroic experimental work, Bose–Einstein condensation of atomic hydrogen was achieved in 1998 [12].

As a consequence of the dramatic advances made in laser cooling of alkali atoms, such atoms became attractive candidates for Bose–Einstein condensation, and they were used in the first successful experiments to produce a gaseous Bose–Einstein condensate. Other atomic species, among them noble gas atoms in excited states, are also under active investigation, and in 2001 two groups produced condensates of metastable ^{4}He atoms in the lowest spin-triplet state [13, 14].

The properties of interacting Bose fluids are treated in many texts. The reader will find an illuminating discussion in the volume by Nozières and Pines [15]. A collection of articles on Bose–Einstein condensation in various systems, prior to its discovery in atomic vapours, is given in [16], while more recent theoretical developments have been reviewed in [17]. The 1998 Varenna lectures describe progress in both experiment and theory on Bose–Einstein condensation in atomic gases, and contain in addition historical accounts of the development of the field [18]. For a tutorial review of some concepts basic to an understanding of Bose–Einstein condensation in dilute gases see Ref. [19].

1.1 Bose–Einstein condensation in atomic clouds

Bosons are particles with integer spin. The wave function for a system of identical bosons is symmetric under interchange of any two particles. Unlike fermions, which have half-odd-integer spin and antisymmetric wave functions, bosons may occupy the same single-particle state. An order-of-magnitude estimate of the transition temperature to the Bose–Einstein condensed state may be made from dimensional arguments. For a uniform gas of free particles, the relevant quantities are the particle mass m, the number density n, and the Planck constant $h = 2\pi\hbar$. The only energy that can be formed from $\hbar$, n, and m is $\hbar^2 n^{2/3}/m$. By dividing this energy by the Boltzmann constant k we obtain an estimate of the condensation temperature T_c,

$$T_c = C\frac{\hbar^2 n^{2/3}}{mk}. \tag{1.1}$$

Here C is a numerical factor which we shall show in the next chapter to be equal to approximately 3.3. When (1.1) is evaluated for the mass and density appropriate to liquid ^{4}He at saturated vapour pressure one obtains a transition temperature of approximately 3.13 K, which is close to the temperature below which superfluid phenomena are observed, the so-called lambda point[2] (T_λ= 2.17 K at saturated vapour pressure).

An equivalent way of relating the transition temperature to the particle density is to compare the thermal de Broglie wavelength λ_T with the

[2] The name lambda point derives from the measured shape of the specific heat as a function of temperature, which near the transition resembles the Greek letter λ.

mean interparticle spacing, which is of order $n^{-1/3}$. The thermal de Broglie wavelength is conventionally defined by

$$\lambda_T = \left(\frac{2\pi\hbar^2}{mkT} \right)^{1/2}. \tag{1.2}$$

At high temperatures, it is small and the gas behaves classically. Bose–Einstein condensation in an ideal gas sets in when the temperature is so low that λ_T is comparable to $n^{-1/3}$. For alkali atoms, the densities achieved range from 10^{13} cm^{-3} in early experiments to 10^{14}–10^{15} cm^{-3} in more recent ones, with transition temperatures in the range from 100 nK to a few μK. For hydrogen, the mass is lower and the transition temperatures are correspondingly higher.

In experiments, gases are non-uniform, since they are contained in a trap, which typically provides a harmonic-oscillator potential. If the number of particles is N, the density of gas in the cloud is of order N/R^3, where the size R of a thermal gas cloud is of order $(kT/m\omega_0^2)^{1/2}$, ω_0 being the angular frequency of single-particle motion in the harmonic-oscillator potential. Substituting the value of the density $n \sim N/R^3$ at $T = T_c$ into Eq. (1.1), one sees that the transition temperature is given by

$$kT_c = C_1 \hbar\omega_0 N^{1/3}, \tag{1.3}$$

where C_1 is a numerical constant which we shall later show to be approximately 0.94. The frequencies for traps used in experiments are typically of order 10^2 Hz, corresponding to $\omega_0 \sim 10^3$ s^{-1}, and therefore, for particle numbers in the range from 10^4 to 10^7, the transition temperatures lie in the range quoted above. Estimates of the transition temperature based on results for a uniform Bose gas are therefore consistent with those for a trapped gas.

In the original experiment [1] the starting point was a room-temperature gas of rubidium atoms, which were trapped and cooled to about 10 μK by bombarding them with photons from laser beams in six directions – front and back, left and right, up and down. Subsequently the lasers were turned off and the atoms trapped magnetically by the Zeeman interaction of the electron spin with an inhomogeneous magnetic field. If we neglect complications caused by the nuclear spin, an atom with its electron spin parallel to the magnetic field is attracted to the minimum of the magnetic field, while one with its electron spin antiparallel to the magnetic field is repelled. The trapping potential was provided by a quadrupole magnetic field, upon which a small oscillating bias field was imposed to prevent loss of particles at the centre of the trap. Some more recent experiments have employed other magnetic field configurations.

In the magnetic trap the cloud of atoms was cooled further by evaporation. The rate of evaporation was enhanced by applying a radio-frequency magnetic field which flipped the electronic spin of the most energetic atoms from up to down. Since the latter atoms are repelled by the trap, they escape, and the average energy of the remaining atoms falls. It is remarkable that no cryogenic apparatus was involved in achieving the record-low temperatures in the experiment [1]. Everything was held at room temperature except the atomic cloud, which was cooled to temperatures of the order of 100 nK.

So far, Bose–Einstein condensation has been realized experimentally in dilute gases of rubidium, sodium, lithium, hydrogen, and metastable helium atoms. Due to the difference in the properties of these atoms and their mutual interaction, the experimental study of the condensates has revealed a range of fascinating phenomena which will be discussed in later chapters. The presence of the nuclear and electronic spin degrees of freedom adds further richness to these systems when compared with liquid ^{4}He, and it gives the possibility of studying multi-component condensates. From a theoretical point of view, much of the appeal of Bose–Einstein condensed atomic clouds stems from the fact that they are dilute in the sense that the scattering length is much less than the interparticle spacing. This makes it possible to calculate the properties of the system with high precision. For a *uniform* dilute gas the relevant theoretical framework was developed in the 1950s and 60s, but the presence of a confining potential – essential to the observation of Bose–Einstein condensation in atomic clouds – gives rise to new features that are absent for uniform systems.

1.2 Superfluid ^{4}He

Many of the concepts used to describe properties of quantum gases were developed in the context of liquid ^{4}He. The helium liquids are exceptions to the rule that liquids solidify when cooled to sufficiently low temperatures, because the low mass of the helium atom makes the zero-point energy large enough to overcome the tendency to crystallization. At the lowest temperatures the helium liquids solidify only under a pressure in excess of 25 bar (2.5 MPa) for ^{4}He and 34 bar for the lighter isotope ^{3}He.

Below the lambda point, liquid ^{4}He becomes a superfluid with many remarkable properties. One of the most striking is the ability to flow through narrow channels without friction. Another is the existence of quantized vorticity, the quantum of circulation being given by h/m $(= 2\pi\hbar/m)$. The occurrence of frictionless flow led Landau and Tisza to introduce a two-fluid description of the hydrodynamics. The two fluids – the normal and the superfluid components – are interpenetrating, and their densities depend on temperature. At very low temperatures the density of the normal component vanishes, while the density of the superfluid component approaches the total density of the liquid. The superfluid density is therefore generally quite different from the density of particles in the condensate, which for liquid ^{4}He is only about 10 % or less of the total, as mentioned above. Near the transition temperature to the normal state the situation is reversed: here the superfluid density tends towards zero as the temperature approaches the lambda point, while the normal density approaches the density of the liquid.

The properties of the normal component may be related to the elementary excitations of the superfluid. The concept of an elementary excitation plays a central role in the description of quantum systems. In an ideal gas an elementary excitation corresponds to the addition of a single particle in a momentum eigenstate. Interactions modify this picture, but for low excitation energies there still exist excitations with well-defined energies. For small momenta the excitations in liquid ^{4}He are sound waves or *phonons*. Their dispersion relation is linear, the energy ϵ being proportional to the magnitude of the momentum p,

$$\epsilon = sp, \tag{1.4}$$

where the constant s is the velocity of sound. For larger values of p, the dispersion relation shows a slight upward curvature for pressures less than 18 bar, and a downward one for higher pressures. At still larger momenta, $\epsilon(p)$ exhibits first a local maximum and subsequently a local minimum. Near this minimum the dispersion relation may be approximated by

$$\epsilon(p) = \Delta + \frac{(p - p_0)^2}{2m^*}, \tag{1.5}$$

where m^* is a constant with the dimension of mass and p_0 is the momentum at the minimum. Excitations with momenta close to p_0 are referred to as *rotons*. The name was coined to suggest the existence of vorticity associated with these excitations, but they should really be considered as short-wavelength phonon-like excitations. Experimentally, one finds at zero pressure that m^* is 0.16 times the mass of a ^{4}He atom, while the constant Δ, the energy gap, is given by $\Delta/k = 8.7$ K. The roton minimum occurs at a wave number $p_0/\hbar$ equal to 1.9×10^8 cm^{-1} (see Fig. 1.1). For excitation

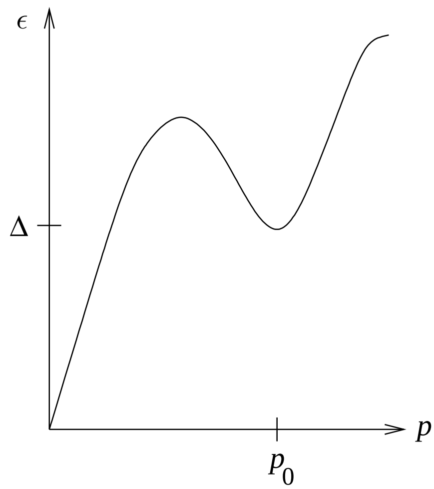

Fig. 1.1. The spectrum of elementary excitations in superfluid ^{4}He. The minimum roton energy is Δ, corresponding to the momentum p_0.

energies greater than 2Δ the excitations become less well-defined since they can decay into two rotons.

The elementary excitations obey Bose statistics, and therefore in thermal equilibrium the distribution function f^0 for the excitations is given by

$$f^0 = \frac{1}{e^{\epsilon(p)/kT} - 1}. \tag{1.6}$$

The absence of a chemical potential in this distribution function is due to the fact that the number of excitations is not a conserved quantity: the energy of an excitation equals the difference between the energy of an excited state and the energy of the ground state for a system containing the same number of particles. The number of excitations therefore depends on the temperature, just as the number of phonons in a solid does. This distribution function Eq. (1.6) may be used to evaluate thermodynamic properties.

1.3 Other condensates

The concept of Bose–Einstein condensation finds applications in many systems other than liquid ^{4}He and the clouds of spin-polarized boson alkali atoms, atomic hydrogen, and metastable helium atoms discussed above. Historically, the first of these were superconducting metals, where the bosons are pairs of electrons with opposite spin. Many aspects of the behaviour of superconductors may be understood qualitatively on the basis of the idea that pairs of electrons form a Bose–Einstein condensate, but the properties of superconductors are quantitatively very different from those of a weakly-interacting gas of pairs. The important physical point is that the binding energy of a pair is small compared with typical atomic energies, and at the temperature where the condensate disappears the pairs themselves break up. This situation is to be contrasted with that for the atomic systems, where the energy required to break up an atom is the ionization energy, which is of order electron volts. This corresponds to temperatures of tens of thousands of degrees, which are much higher than the temperatures for Bose–Einstein condensation.

Many properties of high-temperature superconductors may be understood in terms of Bose–Einstein condensation of pairs, in this case of holes rather than electrons, in states having predominantly d-like symmetry in contrast to the s-like symmetry of pairs in conventional metallic superconductors. The rich variety of magnetic and other behaviour of the superfluid phases of liquid ^{3}He is again due to condensation of pairs of fermions, in this case ^{3}He atoms in triplet spin states with p-wave symmetry. Considerable experimental effort has been directed towards creating Bose–Einstein condensates of excitons, which are bound states of an electron and a hole [20], and of biexcitons, molecules made up of two excitons [21].

Bose–Einstein condensation of pairs of fermions is also observed experimentally in atomic nuclei, where the effects of neutron–neutron, proton–proton, and neutron–proton pairing may be seen in the excitation spectrum as well as in reduced moments of inertia. A significant difference between nuclei and superconductors is that the size of a pair in bulk nuclear matter is large compared with the nuclear size, and consequently the manifestations of Bose–Einstein condensation in nuclei are less dramatic than they are in bulk systems. Theoretically, Bose–Einstein condensation of nucleon pairs is expected to play an important role in the interiors of neutron stars, and observations of glitches in the spin-down rate of pulsars have been interpreted in terms of neutron superfluidity. The possibility of mesons, either pions or kaons, forming a Bose–Einstein condensate in the cores of neutron stars has been widely discussed, since this would have far-reaching consequences for theories of supernovae and the evolution of neutron stars [22].

In the field of nuclear and particle physics the ideas of Bose–Einstein condensation also find application in the understanding of the vacuum as a condensate of quark–antiquark ($\mathrm{u\bar{u}}$, $\mathrm{d\bar{d}}$ and $\mathrm{s\bar{s}}$) pairs, the so-called chiral condensate. This condensate gives rise to particle masses in much the same way as the condensate of electron pairs in a superconductor gives rise to the gap in the electronic excitation spectrum.

This brief account of the rich variety of contexts in which the physics of Bose–Einstein condensation plays a role, shows that an understanding of the phenomenon is of importance in many branches of physics.

1.4 Overview

To assist the reader, we give here a 'road map' of the material we cover. We begin, in Chapter 2, by discussing Bose–Einstein condensation for non-interacting gases in a confining potential. This is useful for developing understanding of the phenomenon of Bose–Einstein condensation and for application to experiment, since in dilute gases many quantities, such as the transition temperature and the condensate fraction, are close to those predicted for a non-interacting gas. We also discuss the density profile and the velocity distribution of an atomic cloud at zero temperature. When the thermal energy kT exceeds the spacing between the energy levels of an atom in the confining potential, the gas may be described semi-classically in terms of a particle distribution function that depends on both position and momentum. We employ the semi-classical approach to calculate thermodynamic quantities. The effect of finite particle number on the transition temperature is estimated, and Bose–Einstein condensation in lower-dimensional systems is discussed.

In experiments to create a Bose–Einstein condensate in a dilute gas the particles used have been primarily alkali atoms and hydrogen, whose spins are non-zero. The new methods to trap and cool atoms that have been developed in recent years make use of the basic atomic structure of these atoms, which is the subject of Chapter 3. There we also study the energy levels of an atom in a static magnetic field, which is a key element in the physics of trapping, and discuss the atomic polarizability in an oscillating electric field.

A major experimental breakthrough that opened up this field was the development of laser cooling techniques. In contrast to so many other proposals which in practice work less well than predicted theoretically, these turned out to be far more effective than originally estimated. Chapter 4 describes magnetic traps, the use of lasers in trapping and cooling, and evaporative cooling, which is the key final stage in experiments to make Bose–Einstein condensates.

In Chapter 5 we consider atomic interactions, which play a crucial role in evaporative cooling and also determine many properties of the condensed state. At low energies, interactions between particles are characterized by the scattering length a, in terms of which the total scattering cross section at low energies is given by $8\pi a^2$ for identical bosons. At first sight, one might expect that, since atomic sizes are typically of order the Bohr radius, scattering lengths would also be of this order. In fact they are one or two orders of magnitude larger for alkali atoms, and we shall show how this may be understood in terms of the long-range part of the interatomic force, which is due to the van der Waals interaction. We also show that the sign of the effective interaction at low energies depends on the details of the short-range part of the interaction. Following that we extend the theory to take into account transitions between channels corresponding to the different hyperfine states for the two atoms. We then estimate rates of inelastic processes, which are a mechanism for loss of atoms from traps, and present the theory of Feshbach resonances, which may be used to tune atomic interactions by varying the magnetic field. Finally we list values of the scattering lengths for the alkali atoms currently under investigation.

The ground-state energy of clouds in a confining potential is the subject of Chapter 6. While the scattering lengths for alkali atoms are large compared with atomic dimensions, they are small compared with atomic separations in gas clouds. As a consequence, the effects of atomic interactions in the ground state may be calculated very reliably by using a pseudopotential proportional to the scattering length. This provides the basis for a mean-field description of the condensate, which leads to the Gross–Pitaevskii equation. From this we calculate the energy using both variational methods and the Thomas–Fermi approximation. When the atom–atom interaction is attractive, the system becomes unstable if the number of particles exceeds a critical value, which we calculate in terms of the trap parameters and the scattering length. We also consider the structure of the condensate at the surface of a cloud, and the characteristic length for healing of the condensate wave function.

In Chapter 7 we discuss the dynamics of the condensate at zero temperature, treating the wave function of the condensate as a classical field. We derive the coupled equations of motion for the condensate density and velocity, and use them to determine the elementary excitations in a uniform gas and in a trapped cloud. We describe methods for calculating collective properties of clouds in traps. These include the Thomas–Fermi approximation and a variational approach based on the idea of collective coordinates. The methods are applied to treat oscillations in both spherically-symmetric and anisotropic traps, and the free expansion of the condensate. We show that, as a result of the combined influence of non-linearity and dispersion, there exist soliton solutions to the equations of motion for a Bose–Einstein condensate.

The microscopic, quantum-mechanical theory of the Bose gas is treated in Chapter 8. We discuss the Bogoliubov approximation and show that it gives the same excitation spectrum as that obtained from classical equations of motion in Chapter 7. At higher temperatures thermal excitations deplete the condensate, and to treat these situations we discuss the Hartree–Fock and Popov approximations. Finally we analyse collisional shifts of spectral lines, such as the 1S–2S two-photon absorption line in spin-polarized hydrogen, which is used experimentally to probe the density of the gas, and lines used as atomic clocks.

One of the characteristic features of a superfluid is its response to rotation, in particular the occurrence of quantized vortices. We discuss in Chapter 9 properties of vortices in atomic clouds and determine the critical angular velocity for a vortex state to be energetically favourable. We also calculate the force on a moving vortex line from general hydrodynamic considerations. The nature of the lowest-energy state for a given angular momentum is considered, and we discuss the weak-coupling limit, in which the interaction energy is small compared with the energy quantum of the harmonic-oscillator potential.

In Chapter 10 we treat some basic aspects of superfluidity. The Landau criterion for the onset of dissipation is discussed, and we introduce the two-fluid picture, in which the condensate and the excitations may be regarded as forming two interpenetrating fluids, each with temperature-dependent densities. We calculate the damping of collective modes in a homogeneous gas at low temperatures, where the dominant process is Landau damping. As an application of the two-fluid picture we derive the dispersion relation for the coupled sound-like modes, which are referred to as first and second sound.

Chapter 11 deals with particles in traps at non-zero temperature. The effects of interactions on the transition temperature and thermodynamic properties are considered. We also discuss the coupled motion of the condensate and the excitations at temperatures below $T_{\rm c}$. We then present calculations for modes above $T_{\rm c}$, both in the hydrodynamic regime, when collisions are frequent, and in the collisionless regime, where we obtain the mode attenuation from the kinetic equation for the particle distribution function.

Chapter 12 discusses properties of mixtures of bosons, either different bosonic isotopes, or different internal states of the same isotope. In the former case, the theory may be developed along lines similar to those for a single-component system. For mixtures of two different internal states of the same isotope, which may be described by a spinor wave function, new possibilities arise because the number of atoms in each state is not conserved. We derive results for the static and dynamic properties of such mixtures. An interesting result is that for an antiferromagnetic interaction between atomic spins, the simple Gross–Pitaevskii treatment fails, and the ground state may be regarded as a Bose–Einstein condensate of *pairs* of atoms, rather than of single atoms.

In Chapter 13 we take up a number of topics related to interference and correlations in Bose–Einstein condensates and applications to matter wave optics. First we describe interference between two Bose–Einstein condensed clouds, and explore the reasons for the appearance of an interference pattern even though the phase difference between the wave functions of particles in the two clouds is not fixed initially. We then demonstrate the suppression of density fluctuations in a Bose–Einstein condensed gas. Following that we consider how properties of coherent matter waves may be investigated by manipulating condensates with lasers. The final section considers the question of how to characterize Bose–Einstein condensation microscopically.

Trapped Fermi gases are considered in Chapter 14. We first show that interactions generally have less effect on static and dynamic properties of fermions than they do for bosons, and we then calculate equilibrium properties of a free Fermi gas in a trap. The interaction can be important if it is attractive, since at sufficiently low temperatures the fermions are then expected to undergo a transition to a superfluid state similar to that for electrons in a metallic superconductor. We derive expressions for the transition temperature and the gap in the excitation spectrum at zero temperature, and we demonstrate that they are suppressed due to the modification of the interaction between two atoms by the presence of other atoms. We also consider how the interaction between fermions is altered by the addition of bosons and show that this can enhance the transition temperature. Finally we briefly describe properties of sound modes in a superfluid Fermi gas, since measurement of collective modes has been proposed as a probe of the transition to a superfluid state.

Problems

PROBLEM 1.1 Consider an ideal gas of ^{87}Rb atoms at zero temperature, confined by the harmonic-oscillator potential

$$V(r) = \frac{1}{2} m\omega_0^2 r^2,$$

where m is the mass of a ^{87}Rb atom. Take the oscillator frequency ω_0 to be given by $\omega_0/2\pi = 150$ Hz, which is a typical value for traps in current use. Determine the ground-state density profile and estimate its width. Find the root-mean-square momentum and velocity of a particle. What is the density at the centre of the trap if there are 10^4 atoms?

PROBLEM 1.2 Determine the density profile for the gas discussed in Problem 1.1 in the classical limit, when the temperature T is much higher than the condensation temperature. Show that the central density may be written as N/R_{th}^3 and determine R_{th}. At what temperature does the mean distance between particles at the centre of the trap become equal to the thermal de Broglie wavelength λ_T? Compare the result with the transition temperature (1.3).

PROBLEM 1.3 Estimate the number of rotons contained in 1 cm^3 of liquid ^{4}He at temperatures $T = 1$ K and $T = 100$ mK at saturated vapour pressure.

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